Through the use of recurrence formulas one can further simplify Eq (9) to yield

$$J_\beta'(x)Y_\beta'(kx) - J_\beta'(kx)Y_\beta'(x) = 0 \tag{10}$$

where $k = b/a$ and $x = a\alpha$ Some of the roots of this equation are given by Truell in Ref 4

Appendix

In Eq (7) let the denominator be called $s\Delta s$, the numerator N, and $B_n \cos\beta\theta \equiv A_n$, for short From the inversion theorem

$$v(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\,t}\bar{v}(s)ds \tag{A1}$$

Let $e^{\,t}\bar{v}(s) = E$ For $\beta \neq 0$, E has a simple pole at $s = 0$ and simple poles at $s = -\kappa\,\alpha^2{}_{m\,\beta}$ where $\alpha_{m\,\beta}$ are the roots (all real and simple) of

$$\begin{aligned}\Delta s = {} & [\beta K_\beta(s^{1/2}b) - s^{1/2}bK_{\beta+1}(s^{1/2}b)] \times \\ & [\beta I_\beta(s^{1/2}a) + s^{1/2}aI_{\beta+1}(s^{1/2}a)] - \\ & [\beta I_\beta(s^{1/2}b) + s^{1/2}bI_{\beta+1}(s^{1/2}b)] \times \\ & [\beta K_\beta(s^{1/2}a) - s^{1/2}aK_{\beta+1}(s^{1/2}a)] = 0\end{aligned} \tag{A2}$$

For $\beta \neq 0$ the residue at the pole $s = 0$ is

$$\lim_{s\to 0}\left\{s\,\frac{N}{s\Delta s}\right\} = \frac{(a/r)^\beta + (r/a)^\beta}{\beta[(b/a)^\beta - (a/b)^\beta]} \tag{A3}$$

where, for $x \to 0$, we have used the relations

$$I_\beta(x) \to (1/2^\beta\beta!)x^\beta$$

$$K_\beta(x) \to 2^{\beta-1}(\beta - 1)!x^{-\beta}\beta \neq 0$$

$$K_0(x) \to -\ln x$$

For $\beta = 0$ the residue at the pole $s = 0$ is

$$\lim_{s\to 0}\frac{d}{ds}\left\{e^{\,t}\,\frac{s^{1/2}aK_0(s^{1/2}r)I_1(s^{1/2}a) - s^{1/2}aI_0(s^{1/2}r)K_1(s^{1/2}a)}{abI_1(s^{1/2}b)K_1(s^{1/2}a) - abI_1(s^{1/2}a)K_1(s^{1/2}b)}\right\} \tag{A4}$$

Let $\mu = s^{1/2}$; then $d/ds = (1/2\mu)(d/d\mu)$ By differentiating Eq (A4) and taking the limit $s \to 0$, one has for the residue of $s = 0$

$$\frac{2bt}{b^2 - a^2} + b\,\frac{(r/a)^2/2 + \ln(r/a) - 1/2}{b^2 - a^2} \tag{A5}$$

To find residue at the pole $s = -\kappa_r\alpha^2{}_{m\,\beta}$, we need

$$s\,\frac{d(\Delta s)}{ds}\bigg|_{s=-\kappa\,\alpha^2{}_{m\,\beta}} = \frac{1}{2}\,\mu\,\frac{d\Delta}{d\mu}\bigg|_{\mu = i\alpha_{m\,\beta}\kappa_r{}^{1/2}} =$$

$$\frac{1}{2}\,i\alpha_{m\,\beta}\kappa_r{}^{1/2}\left[\begin{aligned} & 2\beta/\mu\{[\beta I_\beta(\mu a) + \mu a I_{\beta+1}(\mu a)] \times \\ & [\beta K_\beta(\mu b) - \mu b K_{\beta+1}(\mu b)] - \\ & [\beta I_\beta(\mu b) + \mu b I_{\beta+1}(\mu b)] \times \\ & [\beta K_\beta(\mu a) - \mu a K_{\beta+1}(\mu a)]\} + \\ & b[\beta I_\beta(\mu a) + \mu a I_{\beta+1}(\mu a)] \times \\ & [\mu b K_\beta(\mu b) + \beta K_{\beta+1}(\mu b)] - \\ & b[b\mu I_\beta(\mu b) - \beta I_{\beta+1}(\mu b)] \times \\ & [\beta K_\beta(\mu a) - \mu a K_{\beta+1}(\mu a)] + \\ & a[\mu a I_\beta(\mu a) - \beta I_{\beta+1}(\mu a)] \times \\ & [\beta K_\beta(\mu b) - \mu b K_{\beta+1}(\mu b)] - \\ & a[\mu a K_\beta(\mu a) + \beta K_{\beta+1}(\mu a)] \times \\ & [\beta I_\beta(\mu b) + \mu b I_{\beta+1}(\mu b)]\end{aligned}\right] \tag{A6}$$

From Eq (A2) one has

$$\frac{\beta K_\beta(\mu b) - \mu b K_{\beta+1}(\mu b)}{\beta K_\beta(\mu a) - \mu a K_{\beta+1}(\mu a)} = \frac{\beta I_\beta(\mu b) + \mu b I_{\beta+1}(\mu b)}{\beta I_\beta(\mu a) + \mu a I_{\beta+1}(\mu a)} = \rho$$

If the foregoing is abbreviated into $m/n = o/l$, respectively, Eq (A6) may be written

$$\begin{aligned} s\,[d(\Delta s)/ds] = {} & \tfrac{1}{2}i\alpha_{m\,\beta}\kappa^{\,1/2}\{bl[\mu b K_\beta(\mu b) + \beta K_{\beta+1}(\mu b)] - \\ & bn[b\mu I_\beta(\mu b) - \beta I_{\beta+1}(\mu b)] + am[\mu a I_\beta(\mu a) - \beta I_{\beta+1}(\mu a)] - \\ & ao[\mu a K_\beta(\mu a) + \beta K_{\beta+1}(\mu a)]\}\end{aligned} \tag{A7}$$

In addition, one has the identities

$$\begin{aligned} I_\beta(x) &= i^{-\beta}J_\beta(ix) & K_\beta(x) &= (\pi/2)i^{\beta+1}H_\beta{}^{(1)}(ix) \\ J_\beta(-y) &= (-1)^\beta J_\beta(y) & H_\beta{}^{(1)}(-y) &= (-1)^\beta H_\beta{}^{(1)}(y)\end{aligned} \tag{A8}$$

Then

$$\rho = \frac{\beta J_\beta(\alpha_{m,\beta}b) - bJ_{\beta+1}(\alpha_{m,\beta}b)}{\beta J_\beta(\alpha_{m\,\beta}a) - aJ_{\beta+1}(\alpha_{m\,\beta}a)}$$

or

$$\left[s\,\frac{d\Delta s}{ds}\right]_{s=-\alpha^2{}_{m\,\beta}} = \frac{1}{2}\rho\,(\beta^2 - \alpha^2{}_{m\,\beta}b^2) - \frac{1}{2\rho}\,(\beta^2 - \alpha^2{}_{m\,\beta}a^2) =$$

$$\frac{M(\alpha_{m,\beta})}{2[\beta J_\beta(\alpha_{m\,\beta}a) - aJ_{\beta+1}(\alpha_{m\,\beta}a)][\beta J_\beta(\alpha_{m\,\beta}b) - bJ_{\beta+1}(\alpha_{m\,\beta}b)]}$$

where

$$\begin{aligned} M(\alpha_{m\,\beta}) = {} & (\beta^2 - \alpha^2{}_{m\,\beta}b^2)[\beta J_\beta(\alpha_{m\,\beta}b) - bJ_{\beta+1}(\alpha_{m\,\beta}b)]^2 - \\ & (\beta^2 - \alpha^2{}_{m\,\beta}a^2)\,[\beta J_\beta(\alpha_{m\,\beta}a) - aJ_{\beta+1}(\alpha_{m\,\beta}a)]^2\end{aligned}$$

References

[1] Giedt, W H and Hornbaker, D R, "Transient temperature variation in a thermally orthotropic plate," ARS J **32**, 1902–1909 (1962)

[2] Carslaw, H S and Jaeger, T C, *Conduction of Heat in Solids* (Clarendon Press, Oxford, England, 1959), 2nd ed, pp 17, 332–334

[3] Erdélyi, A, "Higher transcendental functions," *Bateman Manuscript Project, California Institute of Technology* (McGraw Hill Book Co, Inc, New York, 1953), Vol 2, pp 57–63

[4] Truell, R, "Concerning the roots of $J_n'(x)N_n'(kx) - J_n'(kx)N_n'(x) = 0$," J Appl Phys **14**, 350–352 (1943)

Effect of Thermal Accommodation on Cylinder and Sphere Drag in Free Molecule Flow

George S Springer*
Massachusetts Institute of Technology, Cambridge, Mass

AND

Steven W Tsai†
Aeronutronic, Division of Philco, Newport Beach, Calif

Nomenclature

C_D	=	drag coefficient
F,G	=	functions defined by Eqs (7) and (8)
I_0, I_1	=	modified Bessel functions of first kind, zero, and first order
M	=	Mach number
Q	=	heat transfer rate
Q^*	=	dimensionless heat transfer rate defined by Eq (6)
S	=	molecular speed ratio
T	=	freestream static temperature
T_R	=	ratio of surface to freestream static temperature
T_w	=	surface temperature
α	=	thermal accommodation coefficient
σ, σ'	=	coefficients of tangential and normal momentum transfer
γ	=	specific heat ratio
ρ	=	density of freestream

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* Assistant Professor Member AIAA
† Senior Scientist Member AIAA

Introduction

THE aerodynamic characteristics of bodies in free molecule flow depend on the Mach number M, on the freestream temperature T, on the heat-transfer rate to the surface Q, and on the surface accommodation coefficients σ, σ', and α [1 2] To simplify the calculations of these characteristics, the following two assumptions are often made: 1) completely diffuse reflection of the incident molecular stream ($\sigma = \sigma' = 1\ 0$); and 2) complete thermal accommodation at the surface ($\alpha = 1\ 0$) The first of these assumptions seems to be in accordance with the experimental results obtained to date, as indicated in Ref 3 Values reported for thermal accommodation coefficients range from 0 01, as measured on extremely clean surfaces, to 1 0, observed on untreated technological surfaces [4] The assumption, therefore, that α is close to unity is valid in the case where no particular attention is paid to the condition of the surface

α is considerably lower than the hypothetical value of unity when the surface is heated at high temperatures, bombarded by high energy particles, or exposed to low pressures for long periods of time Thus, the value of α to be used in a given application must be determined experimentally for the problem in question The effect of α on the free molecule aerodynamic coefficients can be evaluated, however, from theoretical considerations The purpose of this study is to determine the influence of the thermal accommodation coefficient on cylinder and sphere drag in free molecule flow These results may be generalized to indicate the effects of α on the aerodynamic coefficients of bodies of different shapes

Method of Analysis

It has been pointed out in Ref 5 that both the thermal accommodation coefficient and the wall temperature T_w may depend on the orientation of the surface In the present analysis it is assumed that α and T_w are constant on the surface and that the gas is monatomic Then, for diffuse reflection, the drag coefficient for a cylinder with its axis normal to the direction of flow may be written as[6]

$$C_D = \frac{\pi^{1/2}e^{-S^2/2}}{S}\left[\left(S^2 + \frac{3}{2}\right) I_0\left(\frac{S^2}{2}\right) + \left(S^2 + \frac{1}{2}\right) I_1\left(\frac{S^2}{2}\right)\right] + \frac{\pi^{3/2}}{4S}(T_R)^{1/2} \quad (1)$$

Similarly, the drag coefficient for a sphere, for diffuse reflection, is

$$C_D = \frac{e^{-S^2}}{\pi^{1/2}S^3}(1 + 2S^2) + \frac{4S^4 + 4S^2 - 1}{2S^4}\operatorname{erf}(S) + \frac{2\pi^{1/2}}{3S}(T_R)^{1/2} \quad (2)$$

In the foregoing expressions, I_0 and I_1 are modified Bessel functions of first kind, and S is the speed ratio defined by

$$S = (\gamma/2)^{1/2}M \quad (3)$$

T_R is the ratio of the wall temperature to the freestream temperature

$$T_R = T_{\mathrm{wall}}/T \quad (4)$$

The wall temperature depends on α and on the heat-transfer rate An equation describing the relationship between these parameters is required, therefore, in order to determine the influence of α on C_D If Q is the total amount of heat removed from the surface per unit area and per unit time, an energy balance gives the following expression, applicable to both cylinders and spheres in free molecule flow,[2 7]

$$T_R = [Q^*/\alpha + F(S)]/G(S) \quad (5)$$

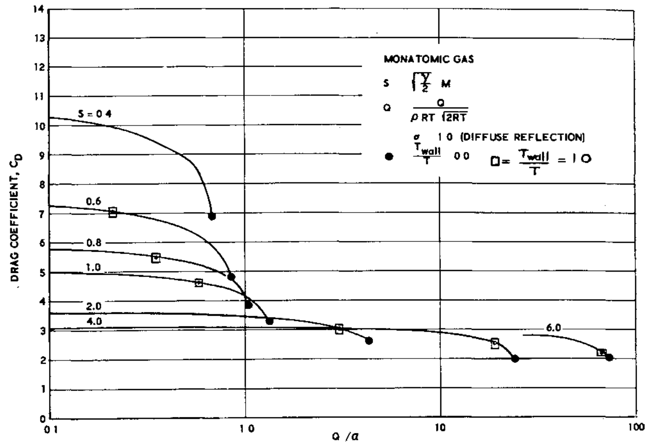

Fig 1 The effect of thermal accommodation on cylinder drag

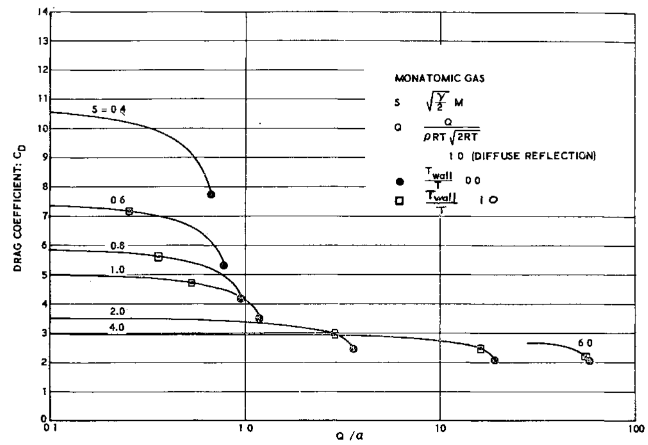

Fig 2 The effect of thermal accommodation on sphere drag

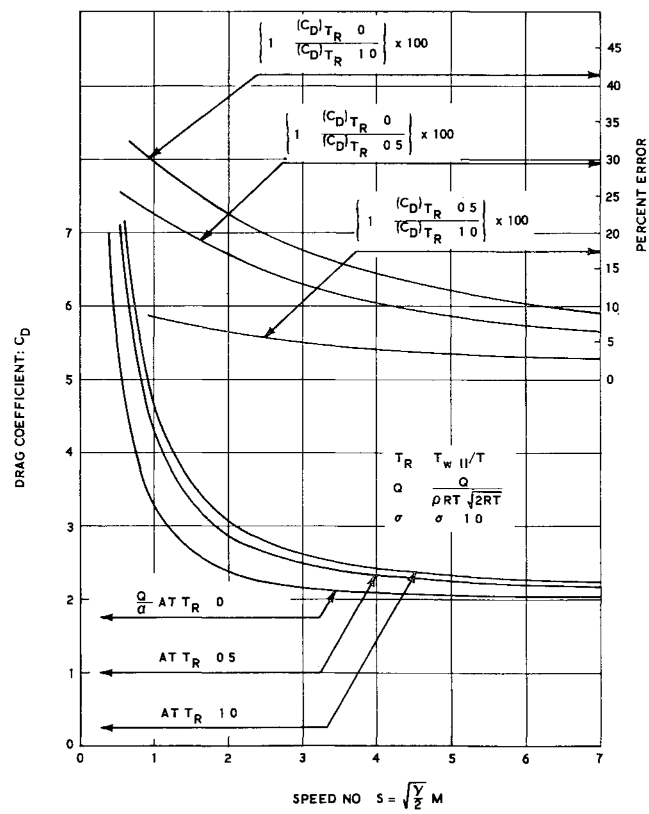

Fig 3 The effect of wall to freestream temperature ratio on cylinder drag

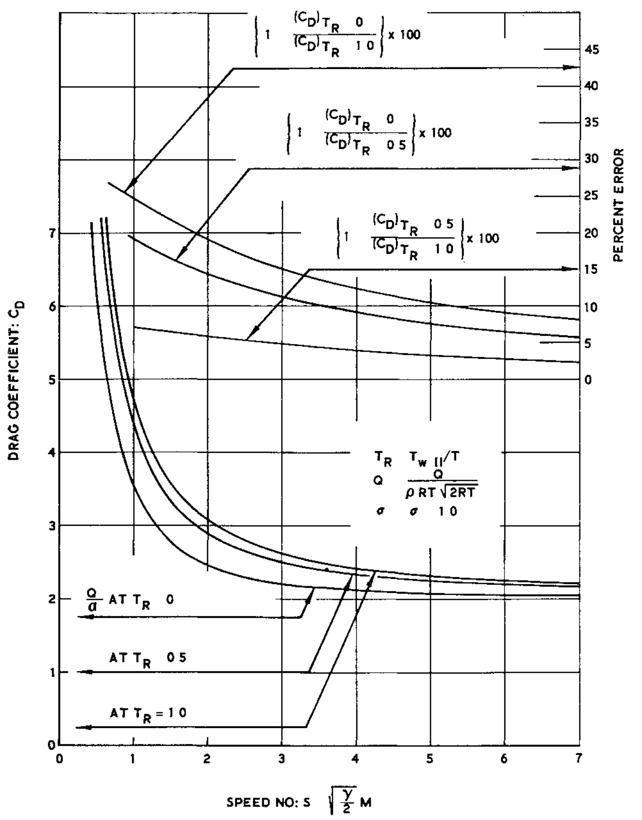

Fig 4 The effect of wall to freestream temperature ratio on sphere drag

where Q^* is a dimensionless heat-transfer parameter,

$$Q^* = Q/[\rho RT(2RT)^{1/2}] \tag{6}$$

The parameters $G(S)$ and $F(S)$ appearing in Eq (5) are functions of the speed ratio For a circular cylinder transverse to the direction of flow,

$$\begin{aligned} G(S) &= \frac{e^{-s^2/2}}{\pi^{1/2}}\left[(S^2+1)I_0\left(\frac{S^2}{2}\right) + S^2 I_1\left(\frac{S^2}{2}\right)\right] \\ F(S) &= \frac{e^{-s^2/2}}{2\pi^{1/2}}\left[\left(S^4 + \frac{7}{2}S^2 + 2\right)I_0\left(\frac{S^2}{2}\right) + \left(S^4 + \frac{5}{2}S^2\right)I_1\left(\frac{S^2}{2}\right)\right] \end{aligned} \tag{7}$$

For a sphere, G and F are

$$\begin{aligned} G(S) &= \frac{S}{2}\left[1 + \frac{1}{S}\,\mathrm{ierfc}(S) + \frac{1}{2S^2}\,\mathrm{erf}(S)\right] \\ F(S) &= \frac{S}{8}\left\{(2S^2+5)\left[1 + \frac{\mathrm{ierfc}(S)}{S}\right] + (2S^2+3)\,\frac{1}{2S^2}\,\mathrm{erf}(S)\right\} \end{aligned} \tag{8}$$

The variation of the drag coefficient with the parameters Q^*/α, T_R, and S may be calculated from the foregoing equations

Results

The effect of the thermal accommodation coefficient on cylinder and sphere drag is shown in Figs 1 and 2 These results indicate that the effect of α on C_D diminishes as the heat-transfer rate approaches zero This is not surprising, since when Q^* is equal to zero the drag coefficient is independent of α, provided that α is not zero In many practical applications the temperature ratio T_R is of the order of 0 3 − 0 5 [5] At these T_R values, a small change in α changes the drag coefficient appreciably for all speed ratios, as can be seen from Figs 1 and 2 Only for high speed ratios and high temperature ratios does the effect of α on C_D become small, a condition seldom encountered in practical problems The results also indicate that for "hypothermal" free molecule flow ($S > 6$), the drag coefficient may be approximated for all values of α by

$$C_D = 2\ 5 \pm 0\ 5 \tag{9}$$

Equation (9) is similar to the one suggested by Schamberg in Ref 8

Q^*/α values corresponding to $T_R = 0$ and $T_R = 1\ 0$ are sometimes used in the calculations of drag coefficients [6 9] The errors due to these assumptions are presented in Figs 3 and 4 It can be seen from these results that the actual T_R values should be used in calculating the drag coefficients, particularly at low speed ratios

References

[1] Bell, S and Schaaf, S A , "Aerodynamic forces on a cylinder for the free molecule flow of a nonuniform gas," ARS J **23**, 314–322 (1953)

[2] Stalder, J R , Goodwin, G , and Craeger, M O , "A comparison of theory and experiment for high speed free molecule flow," NACA Rept 1032 (1951)

[3] Stickney, R E , "An experimental investigation of free molecule momentum transfer between gases and metallic surfaces," Univ Calif , Inst Eng Res , Tech Rept HE-150-182 (1962)

[4] Wachman, H Y , "The thermal accommodation coefficient: a critical survey,"ARS J **32**, 2–12 (1962)

[5] Sentman, L H , "Effect of degree of thermal accommodation on free molecule aerodynamic coefficients," ARS J **32**, 1408–1410 (1962)

[6] Schaaf, S A and Chambré, P L , *Flow of Rarefied Gases* (Princeton University Press, Princeton, N J , 1961) Sec H 8

[7] Sauer, F M , "Convective heat transfer from spheres in a free molecule flow," J Aeronaut Sci **18**, 353–354 (1951)

[8] Schamberg, R , "Analytic representation of surface interaction for free molecule flow with application to drag of various bodies," Rand Corp Rept R-339, Sec 12 (June 1959)

[9] Stalder, J F and Zurick, V J , "Theoretical aerodynamic characteristics of bodies in a free molecule flow field," NACA TN 2423 (1951)

Initial Development of the Laminar Separated Shear Layer

Eric Baum*
Electro-Optical Systems, Inc , Pasadena, Calif

Introduction

SOME recent analytical studies of the compressible laminar shear layer have been based on the physical model pictured in Fig 1 [1 2] The body boundary layer, separating at the rear of the body, becomes the beginning of the separated shear layer, which subsequently grows by entraining gas from the outside inviscid flow ($u = u_e$, $H = H_e$) and from the low-speed base flow ($u = 0$, $H = H_c$) The process of separation is characterized in this model by a discontinuous change in the inside boundary conditions from $u = 0$, $H =$

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* Senior Scientist, Gas Dynamics Department, Fluid Physics Division Member AIAA